Solution
Sheet 1

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Exercise 1

Signature and algebra

- **Signature** (syntactic domain)
  $\Sigma = <SN, FN, domN, ranN>$
  » $SN = \{sn_1, ..., sn_k\}$ set of sort names
  » $FN = \{fn_1, ..., fn_m\}$ set of function names
  » $domN: FN \rightarrow SN^*$ domain
  » $ranN: FN \rightarrow SN$ range
- **Algebra** (semantic domain)
  $A = <S, F, dom, ran>$
  » $S = \{S_1, ..., S_k\}$ set of sorts
  » $F = \{f_1, ..., f_m\}$ set of functions
  » $dom: F \rightarrow S^*$ domain
  » $ran: F \rightarrow S$ range

Signature for integers

- **Signature** (syntactic domain)
  Σ = <*SN, FN, domN, ranN*>
  - » *SN* = {Z} set of sort names
  - » *FN* = {zero, succ, pre, add} set of function names
  - » *domN*: {zero $\rightarrow \varepsilon$, succ $\rightarrow$ Z, pre $\rightarrow$ Z, add $\rightarrow$ Z Z}
  - » *ranN*: {zero $\rightarrow$ Z, succ $\rightarrow$ Z, pre $\rightarrow$ Z, add $\rightarrow$ Z}

Word algebra for integers

- **Algebra** (semantic domain)
  A = <*S, F, dom, ran*>
  - » $S = \{S_Z\}$, where S_Z contains the string representation of terms
  - » $F = \{f_{zero}, f_{succ}, f_{pre}, f_{add}\}$ set of functions
  - » dom: $\{f_{zero} \rightarrow \varepsilon, f_{succ} \rightarrow S_Z, f_{pre} \rightarrow S_Z, f_{add} \rightarrow S_Z\ S_Z\}$
  - » ran: $\{f_{zero} \rightarrow S_Z, f_{succ} \rightarrow S_Z, f_{pre} \rightarrow S_Z, f_{add} \rightarrow S_Z\}$
  - » Functions f_{zero}, f_{succ}, f_{pre}, f_{add} ("&" denotes string concatenation):
    - ⇨ f_{zero} = "zero"
    - ⇨ $f_{succ}(s)$ = "succ(" & s & ")"
    - ⇨ $f_{pre}(s)$ = "pre(" & s & ")"
    - ⇨ $f_{add}(s_1, s_2)$ = "add(" & s_1 & "," & s_2 & ")"

Denotation

- **Denotation** (mapping syntactic $\rightarrow$ semantic domain)
  $\delta : \Sigma \rightarrow A$
  - » δ: $sn_i \rightarrow S_j$ (δ maps each sort name into a sort)
  - » δ: $fn_i \rightarrow F_j$ (analogously for function names)
  - » $dom(\delta(fn_i)) = \delta(domN(fn_i))$, $ran(\delta(fn_i)) = \delta(ranN(fn_i))$
    (domain and range "are preserved")
- **Denotation for integers**
  $\delta : \Sigma \rightarrow A$
  - » δ: {Z $\rightarrow$ S_Z}
  - » δ: {zero $\rightarrow$ f_{zero}, succ $\rightarrow$ f_{succ}, pre $\rightarrow$ f_{pre}, add $\rightarrow$ f_{add}}

Construction of the initial algebra

- **Quotient algebra** of the word algebra:
  - » Subsume all words representing equal terms in an equivalence class
- **Quotient algebra** of the word algebra of integers:
  - » Each element of the quotient algebra corresponds 1:1 to an integer number:
    - ⇨ $S_{QA} = \{s_0, s_{+1}, s_{-1}, s_{+2}, s_{-2}, \ldots\}$
    - ⇨ s_0 = {"zero", "pre(succ(zero))", "succ(pre(zero))", "add(zero,zero)", "add(pre(zero),succ(zero))", "add(succ(zero)),pre(zero))", ...}

Construction of the final algebra

- The **final algebra** is the “smallest” algebra satisfying the presentation
- **Satisfies-Relation**
  Let (Σ, *E*) be a presentation, *A* be an algebra and $\delta : \Sigma \rightarrow A$ be a denotation. *A* satisfies the presentation (Σ, *E*) if and only if:
  - » $t_1 == t_2 \Rightarrow \delta(t_1) = \delta(t_2)$ for all ground substitutions of variables (*)
- **Final algebra for integers** *FA* = <*S, F, dom, ran*>
  - » $S = \{0\}$
  - » $F = \{f_{zero}, f_{succ}, f_{pre}, f_{add}\}$
    - ⇨ $f_{zero} = 0$
    - ⇨ $f_{succ}(0) = 0$
    - ⇨ $f_{pre}(0) = 0$
    - ⇨ $f_{add}(0, 0) = 0$
- (*) is satisfied in a trivial way because *S* contains only one element
- Similarly, there is a trivial homomorphism h: $QA \rightarrow FA$